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On the dimension of the space of $\mathbb{R}$-places of certain rational function fields

Research Article

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Abstract: We prove that for every $n \in \mathbb{N}$ the space $M(K(x_1,\dots,x_n))$ of $\mathbb{R}$-places of the field $K(x_1,\dots,x_n)$ of rational functions of n variables with coefficients in a totally Archimedean field K has the topological covering dimension $\dim M(K(x_1,\dots,x_n)) \le n$. For $n=2$ the space $M(K(x_1,x_2))$ has covering and integral dimensions $\dim M(K(x_1,x_2)) = \dim_{\mathbb{Z}} M(K(x_1,x_2)) = 2$ and the cohomological dimension $\dim_G M(K(x_1,x_2)) = 1$ for any Abelian 2-divisible coefficient group G.

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1. Introduction

In this paper we study the topological structure and evaluate the dimensions of the spaces of $\mathbb{R}$-places of a field K and of its transcendental extensions $K(x_1,\dots,x_n)$ consisting of rational functions of n variables with coefficients in the field K. By the *topological dimension* we understand here the topological covering dimension. The shortest possible way to introduce $\mathbb{R}$-*places* on a field K is to define them as functions $\chi\colon K \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to the extended real line,

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preserving arithmetic operations in the sense that $\chi(0) = 0$, $\chi(1) = 1$, $\chi(x+y) \in \chi(x) \oplus \chi(y)$ and $\chi(x \cdot y) \in \chi(x) \odot \chi(y)$ for all $x, y \in K$, where $\oplus$ and $\odot$ are multivalued extensions of the addition and multiplication operations from $\mathbb{R}$ to $\overline{\mathbb{R}}$. By definition, for $r, s \in \overline{\mathbb{R}}$, $r \oplus s = \{r+s\}$ if $r+s \in \overline{\mathbb{R}}$ is defined and $r \oplus s = \overline{\mathbb{R}}$ if $r+s$ is not defined, i.e. if $r = s = \infty$, in which case $\infty \oplus \infty = \overline{\mathbb{R}}$. By analogy, we define $r \odot s$: it equals the singleton $\{r \cdot s\}$ if $r \cdot s$ is defined and $\overline{\mathbb{R}}$ otherwise, i.e. if $\{r, s\} = \{0, \infty\}$.

Historically, $\mathbb{R}$-places appeared when studying ordered fields. By an *ordered field* we understand a pair (K, P) consisting of a field K and a subset $P \subset K$ called the *positive cone* of (K, P) such that P is an additively closed subgroup of index 2 of the multiplicative group of K. There is a bijective correspondence between positive cones of K and linear orders compatible with addition and multiplication by positive elements. The set $\{a \in K : a > 0\}$ is a positive cone, and the positive cone P generates a total order $<$ on K defined by $x < y$ iff $y - x \in P$. Each ordered field (K, P) has characteristic zero and hence contains the field $\mathbb{Q}$ of rational numbers as a subfield. This fact allows us to define the *Archimedean part*

$$A_P(K) = \big\{x \in K : \text{there exist } a, b \in \mathbb{Q} \text{ such that } a < x < b\big\}$$

of the ordered field (K, P) and also to define the canonical $\mathbb{R}$-place $\chi_P\colon K \to \overline{\mathbb{R}}$ on K assigning $\chi_P(x) = \infty$ to each $x \in K \setminus A_P(K)$ and

$$\chi_P(x) = \sup\{a \in \mathbb{Q} : a \le x\} = \inf\{b \in \mathbb{Q} : b \ge x\} \in \mathbb{R}$$

to each $x \in A_P(K)$. Here the supremum and infimum are taken in the ordered field $\mathbb{R}$ of real numbers.

According to [13, Theorems 1 and 6], a field K admits an $\mathbb{R}$-place if and only if it is *orderable* in the sense that it admits a total order. By [4], each $\mathbb{R}$-place $\chi\colon K \to \overline{\mathbb{R}}$ on a field K is generated by a suitable total order P on K. For an orderable field K denote by $\mathcal{X}(K)$ the space of total orders on K and by $M(K)$ the space of $\mathbb{R}$-places on K. The mentioned results [13] and [4] imply that the map

$$\lambda\colon \mathcal{X}(K) \to M(K), \qquad \lambda\colon P \mapsto \chi_P,$$

assigning to each total order P on K the corresponding $\mathbb{R}$-place χ_P is surjective. The spaces $\mathcal{X}(K)$ and $M(K)$ carry natural compact Hausdorff topologies. Namely, $\mathcal{X}(K)$ carries the Harrison topology generated by the subbase consisting of the sets $a^+ = \{P \in \mathcal{X}(K) : a \in P\}$ where $a \in K \setminus \{0\}$. According to [9, 6.1], the space $\mathcal{X}(K)$ endowed with the Harrison topology is compact Hausdorff and zero-dimensional. By [6], each compact Hausdorff zero-dimensional space is homeomorphic to the space of orderings $\mathcal{X}(K)$ of some field K.

To introduce a natural topology on the space $M(K)$ of $\mathbb{R}$-places of a field K, first endow the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with the topology of one-point compactification of the real line $\mathbb{R}$. It follows from the definition of $\mathbb{R}$-places that the space $M(K)$ is a closed subspace of the compact Hausdorff space $\overline{\mathbb{R}}^K$ of all functions from K to $\overline{\mathbb{R}}$, endowed with the topology of Tychonoff product of the circles $\overline{\mathbb{R}}$. So, $M(K)$ is a compact Hausdorff space, being a closed subspace of the compact Hausdorff space $\overline{\mathbb{R}}^K$.

It turns out that the topology induced on $M(K)$ by the product topology coincides with the quotient topology induced by the mapping $\lambda\colon \mathcal{X}(K) \to M(K)$. This can be seen as follows. The sets

$$U(a) = \{\chi \in M(K) : \chi(a) \in (0, \infty)\}, \qquad a \in K,$$

compose a sub-basis of the quotient topology on $M(K)$. Since these sets are open in the product topology of $M(K)$, the quotient topology is weaker than the product topology. Since the quotient topology is Hausdorff (see [12, Corollary 9.9]) and the product topology is compact (so the weakest among Hausdorff topologies), both topologies on $M(K)$ coincide.

The space $M(K) \subset \overline{\mathbb{R}}^K$ is metrizable if the field K is countable. The converse statement is not true as the uncountable field $\mathbb{R}$ has trivial space of $\mathbb{R}$-places $M(\mathbb{R}) = \{\mathrm{id}\}$. The space of $\mathbb{R}$-places $M(\mathbb{R}(x))$ of the field $\mathbb{R}(x)$ is homeomorphic to the projective line $\overline{\mathbb{R}}$ while $M(\mathbb{R}(x, y))$ is not metrizable, see [15].

In this paper we shall address the following general problem posed in [2].

Problem 1.1.
Investigate the interplay between algebraic properties of a field K and topological properties of its space of $\mathbb{R}$-places $M(K)$.

We shall be mainly interested in the fields $K(x_1, \dots, x_n)$ of rational functions of n variables with coefficients in a subfield $K \subset \mathbb{R}$. It is known that a field K is isomorphic to a subfield of $\mathbb{R}$ if and only if K admits an Archimedean order, i.e., a total order P whose Archimedean part $A_P(K)$ coincides with K. This happens if and only if the corresponding $\mathbb{R}$-place $\chi_P\colon K \to \overline{\mathbb{R}}$ is injective if and only if $\chi_P(K) \subset \mathbb{R}$. By $M_A(K)$ we denote the space of injective $\mathbb{R}$-places on K. Observe that $M_A(K)$ coincides with the space of homomorphisms from K to the real line $\mathbb{R}$.

A field K will be called *totally Archimedean* if it is orderable and each total order on K is Archimedean. Such fields were introduced and characterized in [17]. Important examples of totally Archimedean fields are the fields $\mathbb{Q}$ and $\mathbb{R}$. For a totally Archimedean field K the quotient map $\lambda\colon \mathcal{X}(K) \to M(K)$ is injective. In this case, the spaces $M(K)$ and $\mathcal{X}(K)$ are homeomorphic and hence the space $M(K)$ is zero-dimensional. For a totally Archimedean field K and every $n \in \mathbb{N}$ the topological dimension of the space of $\mathbb{R}$-places $M(K(x_1, \dots, x_n))$ is bounded from above by the transcendence degree n of the field $K(x_1, \dots, x_n)$, as shown in the following theorem which will be proved in Corollary 4.4.

Theorem 1.2.
For any totally Archimedean field K and every $n \in \mathbb{N}$ the space $M(K(x_1, \dots, x_n))$ of $\mathbb{R}$-places of the field $K(x_1, \dots, x_n)$ has topological dimension $\dim M(K(x_1, \dots, x_n)) \le n$.

This theorem suggests the following conjecture.

Conjecture 1.3.
For every totally Archimedean field K and $n \in \mathbb{N}$, $M(K(x_1, \dots, x_n))$ has topological dimension $\dim M(K(x_1, \dots, x_n)) = n$.

For $n = 1$ this conjecture was confirmed (in a stronger form) in [11]: $\dim M(K(x)) = 1$ for any (not necessarily totally Archimedean) real closed field K. The main result of this paper is the following theorem confirming Conjecture 1.3 for $n \le 2$.

Theorem 1.4.
For any field K admitting an Archimedean order we get $\dim M(K(x)) \ge 1$ and $\dim M(K(x, y)) \ge 2$. If the field K is totally Archimedean, then $\dim M(K(x)) = 1$ and $\dim M(K(x, y)) = 2$.

Actually, Theorem 1.4 does not say all the truth about the dimension of the space $M(K(x, y))$. It turns out that this space has covering topological dimension 2 but for any 2-divisible group G the cohomological dimension $\dim_G M(K(x, y))$ is equal to 1! So, the space $M(K(x, y))$ is a natural example of a compact space that is not dimensionally full-valued (which means that the cohomological dimensions of $M(K(x, y))$ for various coefficient groups G do not coincide). A classical example of such a space is the Pontryagin surface, that is a surface with Möbius bands glued at each point of a countable dense subset, see [7, 1.9].

The covering and cohomological dimensions are partial cases of the extension dimension defined as follows, see [8]. We say that the *extension dimension* of a topological space X does not exceed a topological space Y and write $\text{e-dim}\, X \le Y$ if each continuous map $f\colon A \to Y$ defined on a closed subspace A of X can be extended to a continuous map $\bar{f}\colon X \to Y$. The classical Hurewicz–Wallman characterization of the covering dimension [10, 1.9.3] says that $\dim X \le n$ for a separable metric space X if and only if $\text{e-dim}\, X \le S^n$, where S^n stands for the n-dimensional sphere. The sphere S^n is an example of a Moore space $M(\mathbb{Z}, n)$ (whose reduced homology groups $\widetilde{H}_k(S^n)$, $k \ne n$, are trivial except for the n-th group $\widetilde{H}_n(S^n)$ which is isomorphic to $\mathbb{Z}$).

For a non-trivial abelian group G the *cohomological dimension* $\dim_G X$ of a compact space X coincides with the smallest non-negative number n such that $\text{e-dim}\, X \le K(G, n)$, where $K(G, n)$ is the Eilenberg–MacLane complex of G (this is a CW-complex having all homotopy groups trivial except for the n-th homotopy group $\pi_n(K(G,n))$ which is isomorphic to G). If no such n exists, then we put $\dim_G X = \infty$. It is known that $\dim_G X \le \dim X$ for each abelian group G and $\dim X = \dim_{\mathbb{Z}} X$ for any finite-dimensional compact space X. On the other hand, the famous Pontryagin surface Π_2 has covering dimension $\dim \Pi_2 = 2$ and cohomological dimension $\dim_G \Pi_2 = 1$ for any 2-divisible abelian group G, see [7, 1.9]. A group G is called 2-*divisible* if for each $x \in G$ there is $y \in G$ with $y^2 = x$. Surprisingly, for any totally Archimedean field K the space $M(K(x,y))$ has the same pathological dimension properties.

Theorem 1.5.

For any totally Archimedean field K the space of $\mathbb{R}$-places $M(K(x,y))$ has integral cohomological dimension $\dim_{\mathbb{Z}} M(K(x,y)) = \dim M(K(x,y)) = 2$ and the cohomological dimension $\dim_G M(K(x,y)) = 1$ for any non-trivial 2-divisible Abelian group G.

Theorems 1.4 and 1.5 will be proved in Section 4 after some preliminary work made in Section 2.

2. Graphoids and spaces of $\mathbb{R}$-places

In this section we shall discuss the interplay between spaces of $\mathbb{R}$-places and graphoids. The notion of a graphoid has topological nature and can be defined for any family $\mathcal{F}$ of partial functions between topological spaces.

By a *partial function* between topological spaces X, Y we understand a continuous function $f\colon \mathrm{dom}(f) \to Y$ defined on a subspace $\mathrm{dom}(f)$ of the space X. Its *graphoid* $\bar\Gamma(f)$ is the closure of its graph $\Gamma(f) = \{(x, f(x)) : x \in \mathrm{dom}(f)\}$ in the Cartesian product $X \times Y$. The graphoid $\bar\Gamma(f)$ determines a multi-valued extension $\bar f\colon X \multimap Y$ of f whose graph $\Gamma(\bar f) = \{(x,y) \in X \times Y : y \in \bar f(x)\}$ coincides with the graphoid $\bar\Gamma(f)$ of f. The multivalued function $\bar f\colon X \multimap Y$ assigns to each point $x \in X$ the closed subset $\bar f(x) = \{y \in Y : (x,y) \in \bar\Gamma(f)\}$ of the space Y.

For a finite family $\mathcal{F}$ of partial functions between topological spaces X, Y we define the graphoid $\bar\Gamma(\mathcal{F})$ of $\mathcal{F}$ as the graphoid of the "vector" function

$$\mathcal{F}\colon \mathrm{dom}(\mathcal{F}) \to Y^{\mathcal{F}}, \qquad \mathcal{F}\colon x \mapsto (f(x))_{f\in\mathcal{F}},$$

defined on the subset $\mathrm{dom}(\mathcal{F}) = \bigcap_{f\in\mathcal{F}} \mathrm{dom}(f)$.

For an arbitrary family $\mathcal{F}$ of partial functions between X and Y we define its graphoid $\bar\Gamma(\mathcal{F})$ as the intersection

$$\bar\Gamma(\mathcal{F}) = \bigcap \left\{ \mathrm{pr}_{\mathcal{E}}^{-1}(\bar\Gamma(\mathcal{E})) : \mathcal{E} \subset \mathcal{F},\ |\mathcal{E}| < \infty \right\} \subset X \times Y^{\mathcal{F}},$$

where for $\mathcal{E} \subset \mathcal{F}$,

$$\mathrm{pr}_{\mathcal{E}}\colon X \times Y^{\mathcal{F}} \to X \times Y^{\mathcal{E}}, \qquad \mathrm{pr}_{\mathcal{E}}\colon (x, (y_f)_{f\in\mathcal{F}}) \mapsto (x, (y_f)_{f\in\mathcal{E}}),$$

denotes the natural projection. The following lemma, describing the structure of the graphoid $\bar\Gamma(\mathcal{F})$, easily follows from the definition of $\bar\Gamma(\mathcal{F})$.

Lemma 2.1.

The graphoid $\bar\Gamma(\mathcal{F})$ consists of all points $(x, (y_f)_{f\in\mathcal{F}}) \in X \times Y^{\mathcal{F}}$ such that for any finite subfamily $\mathcal{E} \subset \mathcal{F}$ and neighborhoods $O(x) \subset X$ and $O(y_f) \subset Y$ of the points x and y_f, $f \in \mathcal{E}$, there is a point $x' \in O(x) \cap \mathrm{dom}(\mathcal{E})$ such that $f(x') \in O(y_f)$ for all $f \in \mathcal{E}$.

Now we consider the graphoids in the context of rational functions of n variables. To shorten notation, we shall denote the n-tuple $(x_1, \dots, x_n)$ by $\vec{x}$. So, $K(\vec{x})$ will denote the field $K(x_1, \dots, x_n)$ of rational functions of n variables with coefficients in a field K.

Observe that each rational function $f \in \mathbb{R}(\vec{x})$, written as an irreducible fraction $f = p/q$ of two polynomials $p, q \in \mathbb{R}(\vec{x})$, can be thought of as a partial function $f\colon \mathrm{dom}(f) \to \overline{\mathbb{R}}$ defined on the open dense subset $\mathrm{dom}(f) = \mathbb{R}^n \setminus \left(p^{-1}(0) \cap q^{-1}(0)\right)$

of the n-dimensional torus $\overline{\mathbb{R}}^n$. Now we see that any family of rational functions $\mathcal{F} \subset \mathbb{R}(\vec{x})$ can be considered as a family of partial functions whose graphoid $\overline{\Gamma}(\mathcal{F}) \subset \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}}$ is a well-defined closed subset of the compact Hausdorff space $\overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}}$.

Observe that for any finite subfamily $\mathcal{F} \subset \mathbb{R}(\vec{x})$ the subset $\mathrm{dom}(\mathcal{F}) = \bigcap_{f\in\mathcal{F}} \mathrm{dom}(f)$ is open and dense in $\mathbb{R}^n$. Thus the graphoid $\overline{\Gamma}(\mathcal{F}) \subset \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}}$ projects surjectively onto the n-torus $\overline{\mathbb{R}}^n$. The same fact is true for any family $\mathcal{F} \subset \mathbb{R}(\vec{x})$: its graphoid $\overline{\Gamma}(\mathcal{F})$ projects surjectively onto the n-torus $\overline{\mathbb{R}}^n$. It turns out that for a subfield $\mathcal{F} \subset \mathbb{R}(\vec{x})$, containing $\mathbb{Q}(\vec{x})$, the graphoid $\overline{\Gamma}(\mathcal{F})$ can be identified with a subspace of the space of $\mathbb{R}$-places $M(\mathcal{F})$.

Theorem 2.2.

Let $\mathcal{F} \supset \mathbb{Q}(\vec{x})$ be a subfield of the field $\mathbb{R}(\vec{x})$.

(i) *Each point $\gamma = (\vec{a}, (y_f)_{f\in\mathcal{F}})$ of the graphoid $\overline{\Gamma}(\mathcal{F}) \subset \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}}$ determines an $\mathbb{R}$-place $\delta_\gamma\colon \mathcal{F} \to \overline{\mathbb{R}}$, $\delta_\gamma\colon f \mapsto y_f$. To each rational function $f \in \mathcal{F}$ this $\mathbb{R}$-place assigns a point $\delta_\gamma(f) \in \bar{f}(\vec{a})$, where $\bar{f}\colon \overline{\mathbb{R}}^n \multimap \overline{\mathbb{R}}$ is the multivalued extension of f whose graph $\Gamma(\bar{f})$ coincides with the graphoid $\overline{\Gamma}(f)$ of f.*

(ii) *The map $\delta\colon \overline{\Gamma}(\mathcal{F}) \to M(\mathcal{F})$, $\delta\colon \gamma \mapsto \delta_\gamma$, is a topological embedding.*

(iii) *If $\mathcal{F} = \mathbb{K}(\vec{x})$ for some subfield $\mathbb{K} \subset \mathbb{R}$, then $\delta(\overline{\Gamma}(\mathcal{F})) = \{\chi \in M(\mathcal{F}) : \chi{\restriction}_{\mathbb{K}} = \mathrm{id}\}$.*

Proof. (i) Fix a point $\gamma = (\vec{a}, (b_f)_{f\in\mathcal{F}}) \in \overline{\Gamma}(\mathcal{F}) \subset \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}}$ and consider the function $\delta_\gamma\colon \mathcal{F} \to \overline{\mathbb{R}}$, $\delta_\gamma\colon f \mapsto b_f$. Given any rational function $f \in \mathcal{F}$ consider its graphoid $\overline{\Gamma}(f)$, which is equal to the closure of its graph $\{(\vec{x}, f(\vec{x})) : \vec{x} \in \mathrm{dom}(f)\}$ in $\overline{\mathbb{R}}^n \times \overline{\mathbb{R}}$. Next, consider the projection

$$\mathrm{pr}_f\colon \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}} \to \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}, \qquad \mathrm{pr}_f\colon (\vec{x}, (y_f)_{f\in\mathcal{F}}) \mapsto (\vec{x}, y_f)$$

and observe that $\mathrm{pr}_f(\gamma) = (\vec{a}, b_f) \in \overline{\Gamma}(f) = \Gamma(\bar{f})$ by the definition of the graphoid $\overline{\Gamma}(\mathcal{F})$. Consequently, $\delta_\gamma(f) = b_f \in \bar{f}(\vec{a})$. In particular, $\delta_\gamma(x_i) \in \bar{x}_i(\vec{a}) = a_i$ for all $i \le n$. Here a_i denotes the i-th coordinate of the vector $\vec{a} = (a_1, \dots, a_n)$. Also for any constant function $c \in \mathcal{F}$ we get $\delta_\gamma(c) = \bar{c}(\vec{a}) = c$. In particular, $\delta_\gamma(0) = 0$ and $\delta_\gamma(1) = 1$.

To show that δ_γ is an $\mathbb{R}$-place on the field $\mathcal{F}$, it remains to check that $\delta_\gamma(f+g) \in \delta_\gamma(f) \oplus \delta_\gamma(g)$ and $\delta_\gamma(f\cdot g) = \delta_\gamma(f) \odot \delta_\gamma(g)$ for any rational functions $f, g \in \mathcal{F}$. Consider the finite subfamily $\mathcal{E} = \{f, g, f+g, f\cdot g\} \subset \mathcal{F}$, its graph

$$\Gamma(\mathcal{E}) = \big\{(\vec{x}, (y_e)_{e\in\mathcal{E}}) \in \mathrm{dom}(\mathcal{E}) \times \mathbb{R}^{\mathcal{E}} : y_e = e(\vec{x}),\ e \in \mathcal{E}\big\}$$

and its graphoid $\overline{\Gamma}(\mathcal{E}) = \overline{\Gamma(\mathcal{E})} \subset \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{E}}$. Observe that for any point $(\vec{x}, (y_e)_{e\in\mathcal{E}}) \in \Gamma(\mathcal{E})$ we get

$$y_{f+g} = (f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x}) = y_f + y_g$$

and similarly $y_{f\cdot g} = y_f \cdot y_g$. Consequently, $\Gamma(\mathcal{E}) \subset \mathbb{R}^n \times Y$, where $Y = \big\{(y_e)_{e\in\mathcal{E}} \in \mathbb{R}^{\mathcal{E}} : y_{f+g} = y_f + y_g,\ y_{f\cdot g} = y_f \cdot y_g\big\}$. Observe that the closure of the set Y in $\overline{\mathbb{R}}^{\mathcal{E}}$ coincides with the subset

$$\overline{Y} = \big\{(y_e)_{e\in\mathcal{E}} \in \overline{\mathbb{R}}^{\mathcal{E}} : y_{f+g} \in y_f \oplus y_g,\ y_{f\cdot g} = y_f \odot y_g\big\}.$$

Consequently, $\mathrm{pr}_{\mathcal{E}}(\overline{\Gamma}(\mathcal{F})) \subset \overline{\Gamma}(\mathcal{E}) \subset \overline{\mathbb{R}}^n \times \overline{Y}$ which implies the desired inclusions:

$$\delta_\gamma(f+g) = b_{f+g} \ \in\ b_f \oplus b_g = \delta_\gamma(f) \oplus \delta_\gamma(g), \qquad \delta_\gamma(f\cdot g) = b_{f\cdot g} \ \in\ b_f \odot b_g = \delta_\gamma(f) \odot \delta_\gamma(g).$$

(ii) It is easy to see that the map $\delta\colon \overline{\Gamma}(\mathcal{F}) \to M(\mathcal{F})$, $\delta\colon \gamma \mapsto \delta_\gamma$, is continuous. Let us show that it is injective. Take two distinct points $\gamma = (\vec{a}, (b_f)_{f\in\mathcal{F}})$ and $\gamma' = (\vec{a}', (b'_f)_{f\in\mathcal{F}})$ in the graphoid $\overline{\Gamma}(\mathcal{F})$. Then either $b_f \ne b'_f$ for some $f \in \mathcal{F}$ or $a_i \ne a'_i$ for some $i \le n$. If $b_f \ne b'_f$ for some f, then $\delta_\gamma(f) = b_f \ne b'_f = \delta_{\gamma'}(f)$ and hence $\delta_\gamma \ne \delta_{\gamma'}$. If $a_i \ne a'_i$ for some $i \le n$, then for the monomial x_i we get $\delta_\gamma(x_i) = x_i(\vec{a}) = a_i \ne a'_i = x_i(\vec{a}') = \delta_{\gamma'}(x_i)$ and again $\delta_\gamma \ne \delta_{\gamma'}$. Therefore, the continuous map $\delta\colon \overline{\Gamma}(\mathcal{F}) \to M(\mathcal{F})$ is injective. Since the space $\overline{\Gamma}(\mathcal{F})$ is compact and $M(\mathcal{F})$ is Hausdorff, the map δ is a topological embedding.

(iii) Assume that $\mathcal{F} = \mathbb{K}(\vec{x})$ for some subfield $\mathbb{K}$ of $\mathbb{R}$. Then the inclusion $\delta\big(\overline{\Gamma}(\mathcal{F})\big) \subset \{\chi \in M(\mathcal{F}) : \chi{\restriction}_{\mathbb{K}} = \mathrm{id}\}$ follows from the statement (i). To prove the reverse inclusion we shall apply the Tarski–Seidenberg Transfer Principle [16]. This principle says that for two real closed extensions R_1, R_2 of an ordered field K, a finite system of inequalities between polynomials with coefficients in K has a solution in R_1 if and only if it has a solution in the field R_2.

Fix an $\mathbb{R}$-place $\chi\colon \mathcal{F} \to \overline{\mathbb{R}}$ such that $\chi{\restriction}_{\mathbb{K}} = \mathrm{id}$. By [4], the $\mathbb{R}$-place χ is induced by some total order P of the field $\mathcal{F}$. Taking into account that $\chi{\restriction}_{\mathbb{K}} = \mathrm{id}$ is the identity $\mathbb{R}$-place on the field $\mathbb{K}$, we conclude that the orders on $\mathbb{K}$ induced from the ordered fields $(\mathcal{F}, P)$ and $(\mathbb{R}, \mathbb{R}_+)$ coincide. Let $\widehat{\mathbb{K}}$ be the relative algebraic closure of $\mathbb{K}$ in the real closed field $\mathbb{R}$ and $\widehat{\mathcal{F}}$ be a real closure of the ordered field $(\mathcal{F}, P)$. The Uniqueness Theorem [14, XI, § 2] for real closures guarantees that $\widehat{\mathbb{K}}$ can be identified with the real closure of $\mathbb{K}$ in the field $\widehat{\mathcal{F}}$. By [13, Theorem 6], the $\mathbb{R}$-place χ extends to a unique $\mathbb{R}$-place $\widehat{\chi}\colon \widehat{\mathcal{F}} \to \overline{\mathbb{R}}$. The $\mathbb{R}$-place $\widehat{\chi}{\restriction}_{\widehat{\mathbb{K}}}$, being a unique $\mathbb{R}$-place on the real closed field $\widehat{\mathbb{K}}$, coincides with the identity $\mathbb{R}$-place $\mathrm{id}\colon \widehat{\mathbb{K}} \to \mathbb{R}$.

For every $i \le n$ let $a_i = \chi(x_i)$, $\vec{a} = (a_1, \dots, a_n)$, and $b_f = \chi(f)$ for $f \in \mathcal{F}$. The inclusion $\chi \in \delta\big(\overline{\Gamma}(\mathcal{F})\big)$ will be proved as soon as we check that the point $\gamma = (\vec{a}, (b_f)_{f\in\mathcal{F}}) \in \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}}$ belongs to the graphoid $\overline{\Gamma}(\mathcal{F})$. This will follow from Lemma 2.1 as soon as for any finite subfamily $\mathcal{E} \subset \mathcal{F}$, a neighborhood $O(\vec{a}) \subset \overline{\mathbb{R}}^n$ of the point $\vec{a} = (a_1, \dots, a_n)$ and neighborhoods $O(b_f) \subset \overline{\mathbb{R}}$ of the points b_f, $f \in \mathcal{E}$, we find a vector $\vec{z} = (z_1, \dots, z_n) \in O(\vec{a}) \cap \mathrm{dom}(\mathcal{E})$ such that $f(\vec{z}) \in O(b_f)$ for all $f \in \mathcal{E}$.

We lose no generality assuming that $\{x_1, \dots, x_n\} \subset \mathcal{E}$ and $\prod_{i=1}^n O(\chi(x_i)) \subset O(\vec{a})$. Also we can assume that for each function $f \in \mathcal{E}$ the neighborhood $O(b_f)$ is of the form

- $]\alpha_f, \beta_f[$ for some rational numbers $\alpha_f < \beta_f$ if $b_f \in \mathbb{R}$, and
- $\overline{\mathbb{R}} \setminus [\alpha_f, \beta_f]$ for some rational numbers $\alpha_f < \beta_f$ if $b_f = \infty$.

Write each rational function $f \in \mathcal{E}$ as an irreducible fraction $f = p_f/q_f$ of two polynomials $p_f, q_f \in \mathbb{K}(\vec{x})$. Replacing the polynomials p_f and q_f by $-p_f$ and $-q_f$, if necessary, we can assume that $q_f > 0$ in the ordered field $\widehat{\mathcal{F}}$. Write the finite set $\mathcal{E}$ as the union $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_0 \cup \mathcal{E}_+$, where

$$\mathcal{E}_0 = \{f \in \mathcal{E} : \chi(f) \in \mathbb{R}\}, \qquad \mathcal{E}_- = \big\{f \in \mathcal{E} : \chi(f) = \infty,\ f < 0 \text{ in } \widehat{\mathcal{F}}\big\}, \qquad \mathcal{E}_+ = \big\{f \in \mathcal{E} : \chi(f) = \infty,\ f > 0 \text{ in } \widehat{\mathcal{F}}\big\}.$$

To each $f \in \mathcal{E}_0$ we shall assign a system of two polynomial inequalities that has a solution in the field $\widehat{\mathcal{F}}$. Observe that the inclusion $b_f \in O(b_f) =]\alpha_f, \beta_f[$ implies that $\alpha_f < \chi(p_f/q_f) = \widehat{\chi}(p_f/q_f) < \beta_f$. Since the $\mathbb{R}$-place $\widehat{\chi}$ is generated by the total order of the real closed field $\widehat{\mathcal{F}}$, these inequalities are equivalent to the inequalities $\alpha_f < p_f/q_f < \beta_f$ holding in the ordered field $\widehat{\mathcal{F}}$. Since $q_f > 0$, the latter inequalities are equivalent to $\alpha_f q_f < p_f < \beta_f q_f$. It follows that the vector $\vec{x} = (x_1, \dots, x_n) \in \widehat{\mathcal{F}}^n$ is a solution of the system

$$\alpha_f q_f(\vec{x}) < p_f(\vec{x}) < \beta_f q_f(\vec{x})$$

in the real closed field $\widehat{\mathcal{F}}$.

Next, consider the case of a function $f \in \mathcal{E}_+$. Since $\widehat{\chi}(p_f/q_f) = \chi(f) = \infty$ and $f > 0$, we get $\beta_f q_f < p_f$ in $\widehat{\mathcal{F}}$ and, hence, the inequality $\beta_f q_f(\vec{x}) < p_f(\vec{x})$ has a solution $\vec{x} = (x_1, \dots, x_n)$ in $\widehat{\mathcal{F}}$. By the same reason, for every $f \in \mathcal{E}_-$ the inequality $p_f(\vec{x}) < \alpha_f q_f(\vec{x})$ has a solution in $\widehat{\mathcal{F}}$. Therefore, the system of the inequalities

$$\begin{cases} q_f(\vec{x}) > 0 & \text{for all} \quad f \in \mathcal{E}, \\ \alpha_f q_f(\vec{x}) < p_f(\vec{x}) < \beta_f q_f(\vec{x}) & \text{for all} \quad f \in \mathcal{E}_0, \\ \beta_f q_f(\vec{x}) < p_f(\vec{x}) & \text{for all} \quad f \in \mathcal{E}_+, \\ p_f(\vec{x}) < \alpha_f q_f(\vec{x}) & \text{for all} \quad f \in \mathcal{E}_-, \end{cases}$$

has a solution $\vec{x} = (x_1, \dots, x_n)$ in the real closed field $\widehat{\mathcal{F}}$. By the Tarski–Seidenberg Transfer Principle [16, 11.2.2], this system has a solution in the real closed field $\widehat{\mathbb{K}} \subset \mathbb{R}$. Using the continuity of the polynomials p_f, q_f, $f \in \mathcal{E}$, we can find a solution $\vec{z}$ of this system in the dense subset $\big(\widehat{\mathbb{K}} \cap \mathrm{dom}(\mathcal{E})\big)^n$ of $\widehat{\mathbb{K}}^n$. The choice of the inequalities from the system guarantees that $\vec{z} \in \prod_{i=1}^n O(\chi(x_i)) \cap \mathrm{dom}(\mathcal{E})^n \subset O(\vec{a}) \cap \mathrm{dom}(\mathcal{E})^n$ and $f(\vec{z}) = p_f(\vec{z})/q_f(\vec{z}) \in O(b_f)$ for all $f \in \mathcal{E}$. □

Theorem 2.2 will help us to analyze the structure of certain fibers of the restriction operator $\rho_K\colon M(K(\vec{x})) \to M(K)$, $\rho_K\colon \chi \mapsto \chi{\restriction}_K$.

Proposition 2.3.

Take any field K with an injective $\mathbb{R}$-place $\varphi\colon K \to \mathbb{R}$. Then the fiber $\rho_K^{-1}(\varphi) \subset M(K(\vec{x}))$ can be identified with the graphoid $\overline{\Gamma}(\mathcal{F})$, where $\mathcal{F} = \mathbb{K}(\vec{x})$ for $\mathbb{K} = \varphi(K) \subset \mathbb{R}$.

Proof. The $\mathbb{R}$-place $\varphi\colon K \to \mathbb{R}$, being injective, is an isomorphism of the fields K and $\mathbb{K}$. This isomorphism extends to a unique isomorphism $\Phi\colon K(\vec{x}) \to \mathbb{K}(\vec{x})$ such that $\Phi(x_i) = x_i$ for all $i \le n$, where $\vec{x} = (x_1, \dots, x_n)$. The isomorphism $\varphi\colon K \to \mathbb{K}$ induces a homeomorphism $M\varphi\colon M(\mathbb{K}) \to M(K)$ which assigns to each $\mathbb{R}$-place $\chi\colon \mathbb{K} \to \overline{\mathbb{R}}$ the $\mathbb{R}$-place $\chi\circ\varphi\colon K \to \overline{\mathbb{R}}$. In the same way the isomorphism Φ induces a homeomorphism $M\Phi\colon M(\mathbb{K}(\vec{x})) \to M(K(\vec{x}))$. Now look at the commutative diagram

$$\begin{tikzcd}
M(K(\vec{x})) \arrow[d, "\rho_K"'] & M(\mathbb{K}(\vec{x})) \arrow[l, "M\Phi"'] \arrow[d, "\rho_{\mathbb{K}}"] & \\
M(K) & M(\mathbb{K}) \arrow[l, "M\varphi"] & \overline{\Gamma}(\mathbb{K}(\vec{x})). \arrow[ul, "\delta"'] \arrow[dl] \\
\{\varphi\} \arrow[u] & \{\mathrm{id}\} \arrow[l] \arrow[u] &
\end{tikzcd}$$

Here $\delta\colon \overline{\Gamma}(\mathbb{K}(\vec{x})) \to M(\mathbb{K}(\vec{x}))$ is the embedding defined in Theorem 2.2, which implies that $\rho_{\mathbb{K}}^{-1}(\mathrm{id}) = \delta\big(\overline{\Gamma}(\mathbb{K}(\vec{x}))\big)$. Since the maps $M\varphi$ and $M\Phi$ are homeomorphisms, we conclude that the composition $M\Phi\circ\delta$ maps homeomorphically the graphoid $\overline{\Gamma}(\mathbb{K}(\vec{x}))$ onto the fiber $\rho_K^{-1}(\varphi)$. $\square$

3. Dimension of graphoids of families of rational functions

Theorem 3.1.

For every $n \in \mathbb{N}$ and a family of rational functions $\mathcal{F} \subset \mathbb{R}(x_1, \dots, x_n)$ the graphoid $\overline{\Gamma}(\mathcal{F})$ has topological dimension $\dim\overline{\Gamma}(\mathcal{F}) \le n$.

Proof. First we consider the case of a finite family $\mathcal{F}$. By the Sum Theorem [10, Theorem 1.5.3] for topological dimension, the inequality $\dim\overline{\Gamma}(\mathcal{F}) \le n$ will be proved as soon as we check that each point $(\vec{x}, \vec{y}) \in \overline{\Gamma}(\mathcal{F}) \subset \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^{\mathcal{F}}$ has a neighborhood $O(\vec{x}, \vec{y})$ of dimension $\dim O(\vec{x}, \vec{y}) \le n$. Applying to the points $\vec{x}$ and $\vec{y}$ and the family $\mathcal{F}$ suitable projective transformations, we can assume that $(\vec{x}, \vec{y})$ belongs to the Euclidean space $\mathbb{R}^n \times \mathbb{R}^{\mathcal{F}}$. Now it suffices to check that the intersection $\overline{\Gamma}(\mathcal{F}) \cap (\mathbb{R}^n \times \mathbb{R}^{\mathcal{F}})$ has dimension $\le n$. For this we shall use some standard facts from the theory of semi-algebraic sets, see [3, 5].

Write $\mathcal{F}$ as $\mathcal{F} = \{f_1, \dots, f_m\}$ where each $f_i = p_i/q_i$ is a rational function written as an irreducible fraction of two polynomials. Observe that the graph $\Gamma(\mathcal{F})$ of the family $\mathcal{F}$ coincides with the semialgebraic set

$$A = \big\{(\vec{x}, (y_1, \dots, y_m)) \in \mathbb{R}^n \times \mathbb{R}^m : q_i(\vec{x}) \neq 0,\ y_i \cdot q_i(\vec{x}) = p_i(\vec{x}),\ i \le m\big\}$$

in $\mathbb{R}^n \times \mathbb{R}^m$, which is homeomorphic to the open subset $\{\vec{x} \in \mathbb{R}^n : q_i(\vec{x}) \neq 0,\ i \le m\}$ of $\mathbb{R}^n$ and hence has dimension $\dim A = n$. By [5, Proposition 1.9], $\dim\overline{A} = \dim A = n$. Since $\overline{A} = \overline{\Gamma}(\mathcal{F}) \cap (\mathbb{R}^n \times \mathbb{R}^{\mathcal{F}})$, we see that $\dim\overline{\Gamma}(\mathcal{F}) \le n$.

If $\mathcal{F}$ is infinite, then the definition of the graphoid $\overline{\Gamma}(\mathcal{F})$ implies that it is the limit of the inverse spectrum

$$\Sigma = \big\{\overline{\Gamma}(\alpha), \mathrm{pr}^{\beta}_{\alpha} : \alpha \subset \beta \subset \mathcal{F},\ |\beta| < \infty\big\}$$

consisting of graphoids of finite subfamilies $\mathcal{E} \subset \mathcal{F}$ and natural projections $\mathrm{pr}^{\beta}_{\alpha}\colon \overline{\Gamma}(\beta) \to \overline{\Gamma}(\alpha)$ of such graphoids. As we have already proved, graphoids of finite subfamilies of $\mathcal{F}$ have dimension $\le n$. By [10, Theorem 3.4.11], the limit space $\overline{\Gamma}(\mathcal{F})$ of the spectrum Σ has dimension $\dim\overline{\Gamma}(\mathcal{F}) \le n$. $\square$

4. Extension dimension of the space $M(K(\vec{x}))$

In this section we shall evaluate the extension dimension of the space of $\mathbb{R}$-places $M(K(\vec{x}))$ of the field $K(\vec{x}) = K(x_1,\dots,x_n)$ of rational functions of n variables with coefficients in an Archimedean field K, i.e., a field which admits an Archimedean order. Every Archimedean field K can be embedded into $\mathbb{R}$.

We shall say that a topological space Y is an *absolute neighborhood extensor for compacta* (briefly, an ANE) if each continuous map $f\colon B \to Y$ defined on a closed subspace B of a compact Hausdorff space X can be extended to a continuous map $\bar{f}\colon A \to Y$ defined on a neighborhood A of B in X. We recall that a topological space X has extension dimension $\operatorname{e-dim} X \le Y$ if each continuous map $f\colon B \to Y$ defined on a closed subspace B of X admits a continuous extension $\bar{f}\colon X \to Y$.

Theorem 4.1.
Take an Archimedean field K. If the space of $\mathbb{R}$-places of the field $K(\vec{x})$ has extension dimension $\operatorname{e-dim} M(K(\vec{x})) \le Y$*, then for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ has extension dimension* $\operatorname{e-dim}\bar{\Gamma}(\mathbb{K}(\vec{x})) \le Y$*. If K is totally Archimedean and Y is an ANE-space, then also the converse holds.*

Proof. Assume that $\operatorname{e-dim} M(K(\vec{x})) \le Y$ for some space Y. Given any subfield $\mathbb{K} \subset \mathbb{R}$, isomorphic to K, we need to check that $\operatorname{e-dim}\bar{\Gamma}(\mathbb{K}(\vec{x})) \le Y$. Fix any isomorphism $\varphi\colon K \to \mathbb{K}$ and observe that it is an injective $\mathbb{R}$-place on K. By Proposition 2.3, the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ of the function family $\mathbb{K}(\vec{x})$ is homeomorphic to a subspace of the space $M(K(\vec{x}))$. Because of that, $\operatorname{e-dim} M(K(\vec{x})) \le Y$ implies $\operatorname{e-dim}\bar{\Gamma}(\mathbb{K}(\vec{x})) \le Y$.

Now suppose that K is a totally Archimedean field and the space Y is an ANE. Assume that for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ has extension dimension $\operatorname{e-dim}\bar{\Gamma}(\mathbb{K}(\vec{x})) \le Y$. Since the field K is totally Archimedean, the quotient map $\lambda\colon \mathcal{X}(K) \to M(K)$ is a homeomorphism (see [2, Remark 2.16]). Since the space $\mathcal{X}(K)$ is zero-dimensional, so is the space $M(K)$.

Now consider the restriction operator $\rho_K\colon M(K(\vec{x})) \to M(K)$, $\rho_K\colon \chi \mapsto \chi{\restriction}_K$. By Proposition 2.3, for each $\mathbb{R}$-place $\varphi \in M_A(K) = M(K)$ the fiber $\rho_K^{-1}(\varphi)$ is homeomorphic to the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ of the family $\mathbb{K}(\vec{x})$ of rational functions of n variables with coefficients in the subfield $\mathbb{K} = \varphi(K)$ of $\mathbb{R}$. Our assumption on the extension dimension of $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ implies that $\operatorname{e-dim}\rho_K^{-1}(\varphi) \le Y$. The following lemma implies that $\operatorname{e-dim} M(K(\vec{x})) \le Y$. $\square$

Lemma 4.2.
Let $\rho\colon X \to Z$ be a continuous map from a compact Hausdorff space X onto a zero-dimensional compact Hausdorff space Z. The space X has extension dimension $\operatorname{e-dim} X \le Y$ *for some ANE-space Y if and only if for each $z \in Z$ the fiber $\rho^{-1}(z)$ has extension dimension* $\operatorname{e-dim}\rho^{-1}(z) \le Y$.

Proof. The "only if" part trivially follows from the definition of extension dimension. To prove the "if" part, assume that each fiber of ρ has extension dimension $\le Y$. To prove that $\operatorname{e-dim} X \le Y$, fix a continuous map $f\colon B \to Y$ defined on a closed subspace B of X. For each point $z \in Z$ consider the fiber $X_z = \rho^{-1}(z) \subset X$ of the map ρ. Since $\operatorname{e-dim} X_z \le Y$, the map $f{\restriction}_{B\cap X_z}$ admits a continuous extension $f_z\colon X_z \to Y$. Consider the map $\tilde{f}_z\colon X_z \cup B \to Y$ defined by $\tilde{f}_z{\restriction}_{X_z} = f_z$ and $\tilde{f}_z{\restriction}_B = f$. Since Y is an ANE-space, the map $\tilde{f}_z$ admits a continuous extension $\bar{f}_z\colon A_z \to Y$ defined on an open neighborhood A_z of $X_z \cup B$ in X. Since the space X is compact and Z is Hausdorff, the map ρ is closed. Consequently, the set $\rho(X\setminus A_z)$ is closed in Z and its complement $O_z = Z\setminus f(X\setminus A_z)$ is an open neighborhood of z in Z. Since the space Z is compact and zero-dimensional, the open cover $\{O_z : z \in Z\}$ of Z can be refined by a finite disjoint open cover $\mathfrak{U}$. For every set $U \in \mathfrak{U}$ choose a point $z \in Z$ with $U \subset O_z$ and put $\bar{f}_U = \bar{f}_z{\restriction}_{\rho^{-1}(U)}$. It follows that the map f_U is a continuous extension of the map $f{\restriction}_{B\cap\rho^{-1}(U)}$. Then the maps $\bar{f}_U$, $U \in \mathfrak{U}$, compose a required continuous extension $\bar{f} = \bigcup_{U\in\mathfrak{U}} \bar{f}_U\colon X \to Y$ of the map f. $\square$

By the Hurewicz–Wallman theorem [10, 1.9.3], a compact Hausdorff space X has covering topological dimension $\dim X \le d$ for some $d \in \omega$ if and only if $\operatorname{e-dim} X \le S^d$, where S^d stands for the d-dimensional sphere. Because of that, Theorem 4.1 implies

Corollary 4.3.
Take an Archimedean field K. If the space of $\mathbb{R}$-places of the field $K(\vec{x})$ has dimension $\dim M(K(\vec{x})) \le d$ for some $d \in \omega$, then for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\overline{\Gamma}(\mathbb{K}(\vec{x}))$ has dimension $\dim \overline{\Gamma}(\mathbb{K}(\vec{x})) \le d$. If K is totally Archimedean, then also the converse holds.

This corollary combined with Theorem 3.1 implies the following corollary announced in the introduction.

Corollary 4.4.
For any totally Archimedean field K and every $n \in \mathbb{N}$ the space of $\mathbb{R}$-places of the field $K(x_1, \dots, x_n)$ has topological dimension $\dim M(K(x_1, \dots, x_n)) \le n$.

Let us recall from [7], that for an Abelian group G a compact Hausdorff space X has cohomological dimension $\dim_G X \le d$ for some $d \in \omega$ if and only if $\text{e-dim}\, X \le K(G, d)$ (see the introduction). This fact combined with Theorem 4.1 implies

Corollary 4.5.
Take an Archimedean field K. If the space of $\mathbb{R}$-places of the field $K(\vec{x})$ has dimension $\dim_G M(K(\vec{x})) \le d$ for some $d \in \omega$ and some Abelian group G, then for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\overline{\Gamma}(\mathbb{K}(\vec{x}))$ has dimension $\dim_G \overline{\Gamma}(\mathbb{K}(\vec{x})) \le d$. If K is totally Archimedean, then also the converse holds.

Now we see that Theorems 1.4 and 1.5 follow from Corollaries 4.3 and 4.5 and the following deep result [1] about the (cohomological) dimension of the graphoids.

Theorem 4.6 (Banakh–Potyatynyk).

(a) *For any non-empty subfamily $\mathcal{F} \subset \mathbb{R}(x)$ the graphoid $\overline{\Gamma}(\mathcal{F})$ is homeomorphic to the extended real line $\overline{\mathbb{R}}$ and, hence, has dimension $\dim \overline{\Gamma}(\mathcal{F}) = 1$.*

(b) *For any non-empty subfamily $\mathcal{F} \subset \mathbb{R}(x, y)$ the graphoid $\overline{\Gamma}(\mathcal{F})$ has dimensions $\dim \overline{\Gamma}(\mathcal{F}) = \dim_{\mathbb{Z}} \overline{\Gamma}(\mathcal{F}) = 2$.*

(c) *For any subfamily $\mathcal{F} \subset \mathbb{R}(x, y)$ containing the rational functions $(x - a)/(y - b)$, $a, b \in \mathbb{Q}$, the graphoid $\overline{\Gamma}(\mathcal{F})$ has cohomological dimension $\dim_G \overline{\Gamma}(\mathcal{F}) = 1$ for any non-trivial 2-divisible Abelian group G.*

In light of Corollaries 4.3 and 4.5 the following problem arises naturally.

Problem 4.7.
Let $\mathbb{K}, \mathbb{F} \subset \mathbb{R}$ be two isomorphic copies of a (totally Archimedean) field K. Are the graphoids $\overline{\Gamma}(\mathbb{K}(x, y))$ and $\overline{\Gamma}(\mathbb{F}(x, y))$ homeomorphic?

Remark 4.8.
In light of this question it is interesting to remark that a totally Archimedean field can have distinct isomorphic copies in $\mathbb{R}$. A suitable example can be constructed as follows. Take the polynomial $f(x) = x^4 - 5x^2 + 2$. This polynomial is irreducible over $\mathbb{Q}$ and has four real roots. The Galois group of f is the dihedral group with eight elements, so the degree of the splitting field of f over $\mathbb{Q}$ is 8. Therefore, for every root α of f there is another root β such that $\beta \notin \mathbb{Q}(\alpha)$. It follows that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic, but not equal, totally Archimedean subfields of $\mathbb{R}$.

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